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The Arithmetic Fourier Transform and Real Neural Networks: Summability by Primes

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1. INTRODUCTION

The Arithmetic Fourier Transform (AFT) is most suitable for the evaluation of Fourier cosine coefficients. The computation is accomplished by use of the Möbius function defined on the positive integers by

(i) $\mu(1) = 1$;

(ii) $\mu(j) = 0$ if there is a prime p such that p^2/j;

(iii) if $j = p_1 p_2 \ldots p_l$, is the prime factorisation of j, and the p's are all distinct, then $\mu(j) = (-1)^l$.

Suppose that a function f is defined on an interval $[0, \pi]$ (or by a change of scale on $[0, L]$) and that it is extended to be even on $[-\pi, \pi]$ and then to be periodic of period 2π. The algorithm requires that $\int_0^{2\pi} f(\theta)d\theta = 0$.

Then under suitable conditions, to be discussed in section two, the Fourier cosine coefficients are given by

$$a_n = \sum_{k=1}^{\infty} \mu(k)S(nk), \tag{1.1}$$

where

$$S(n) = \frac{1}{n}\sum_{m=0}^{n-1} f\left(\frac{2\pi m}{n}\right). \tag{1.2}$$

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The AFT first aroused interest in the field of signal processing because it was suitable for parallel processing. The drawback is that a signal is usually sampled at regularly spaced values whereas the algorithm requires evaluation of f at irregularly spaced points. This problem was first addressed in [8] where the authors used first order (linear) and zero order (nearest neighbour) interpolation to evaluate f at intermediate points.

Another feaure of the AFT algorithm is that it models the structure of a double layer in an artificial neural network. This raises an interesting question discussed at some length in [15]. If the brain (a real neural network) carries out Fourier analysis, is this similar to the AFT rather than conventional Fourier analysis? The purpose of this paper is to present some mathematical results related to the question.

The first development is a summability method for the algorithm. This is discussed from a numerical point of view in [14] and goes some way towards eliminating the necessity for interpolation between regularly spaced sample points. In Section 2 we discuss the summability method in the general context of Möbius inversion and compare it to the first rigorous treatment of the AFT by Wintner [17].

The second development concerns the variability of the algorithm. The standard algorithm may be extended to a family of algorithms obtained by the deletion of sample points. This result, which is discussed in section three, can be deduced from Vinogradov's work on the theory of primes as developed by Davenport [2]. The objective of section four will be to extend the deletion procedure to the summability algorithm. The variability obtained could be considered as a paradigm for the randomness of synaptic connections in a real neural network. For a further discussion see [15].

2. The Arithmetic Fourier Transform

All versions of the AFT depend on the following two lemmas. The proof of the first is straightforward.

LEMMA 1. *If ω is an nth root of unity then*

$$\sum_{m=0}^{n-1} \omega^{jm} = \begin{cases} 0 & \text{if } j \neq kn \\ n & \text{if } j = kn \end{cases}$$

for some positive integer k.

LEMMA 2. *Let f be an even function of period 2π with Fourier cosine series*

$$f(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta$$

which converges everywhere to f. If for each positive integer n

$$S(n) = \frac{1}{n} \sum_{m=0}^{n-1} f\left(\frac{2\pi m}{n}\right),$$

then

$$S(n) = \sum_{k=1}^{\infty} a_{kn}. \tag{2.3}$$

Proof. We rewrite the series so that

$$f(\theta) = \sum_{j=1}^{\infty} \frac{1}{2} a_j (e^{ij\theta} + e^{-ij\theta}).$$

Then by Lemma 1,

$$S(n) = \frac{1}{2n} \sum_{j=1}^{\infty} a_j \sum_{m=0}^{n-1} (e^{ij2\pi m/n} + e^{-ij2\pi m/n})$$

$$= \sum_{k=1}^{\infty} a_{kn}.$$

The fundamental problem is to invert the system of equations (2.3) to obtain an explicit formula for each a_n. We shall see that there are two ways to do this. The first is well known and depends on the Möbius formula given by

$$\sum_{d/r} \mu(d) = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{if } r \neq 1. \end{cases}$$

The standard AFT formula can be obtained formally by the Möbius inversion,

$$\sum_{k=1}^{\infty} \mu(k) S(nk) = \sum_{k=1}^{\infty} \mu(k) \sum_{j=1}^{\infty} a_{nkj},$$

$$= \sum_{r=1}^{\infty} a_{nr} \sum_{d/r} \mu(d),$$

$$= a_n.$$

The rearrangement of the double series is valued if $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mu(k)a_{nkj}$ converges absolutely. This formal proof was known to Bruns [1] who claimed that the absolute convergence $\sum_{n=1}^{\infty} |a_n| < \infty$ was a sufficient condition for the rearrangement of the double series. This was refuted by Wintner [17], who gave an example to show that, due to the accumulation of divisors, the absolute convergence $\sum_{n=1}^{\infty} |a_n| < \infty$ need not imply the validity of Möbius inversion. However Wintner [17, p. 14] was able to use the results of Hardy [5] to obtain the following two independent sets of sufficient conditions for Möbius inversion.

(i) f is of bounded variation on $[0, 2\pi]$ and $f \in \mathrm{Lip}_\alpha[0, 2\pi]$, $0 < \alpha \leq 1$

(ii) $f \in \mathrm{Lip}_\alpha[0, 2\pi]$, $\frac{1}{2} < \alpha \leq 1$. Here $f \in \mathrm{Lip}_\alpha[0, 2\pi]$ is the usual condition that $|f(\theta_1) - f(\theta_2)| < M|\theta_1 - \theta_2|^\alpha$ for some constant M and all θ_1 and θ_2.

Both sets of conditions (i) and (ii) imply that the Fourier cosine series of f converges absolutely on $[0, 2\pi]$ (See [7, p. 32]). At this point it is appropriate to recall the theorem of Lusin that the absolute convergence of $\sum_{n=1}^{\infty} a_n \cos n\theta$ on any subset of $[0, 2\pi]$ of positive measure is equivalent to $\sum_{n=1}^{\infty} |a_n| < \infty$.

Several authors interested in signal processing and sampling theory ([9–11, 13, 16]) have rediscovered the standard AFT algorithm and proved it under stronger assumptions which were adequate for their purposes. The standard algorithm has also been proved for step functions and hence for the moving average of a Lebesgue integrable function, [12].

As mentioned earlier there is a second approach to the inversion of the system of Eq. (2.3). This is the method of summability by primes which has its roots in the work of Duffin [4], who was concerned with the representation of Fourier integrals. In contrast to Möbius inversion it is sufficient to assume the absolute convergence of the Fourier cosine series. In order to state the summability theorem we need to define the notation δ_j^q for j and q positive integers by

$\delta_j^q = 0$, if j contains a prime factor greater than the qth prime;
$\delta_j^q = 1$, otherwise.

THEOREM 1. *Let f be an even (possibly complex-valued) function of period 2π which is normalised so that $\int_0^{2\pi} f(\theta)d\theta = 0$. Suppose that the Fourier series $f(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta$ of f converges absolutely. Then*

$$a_n = \lim_{q\to\infty} \sum_{j=1}^{\infty} \delta_j^q \mu(j) S(jn), \qquad n \geq 1,$$

where $S(n) = 1/n \sum_{m=0}^{n-1} f(2\pi m/n)$.

Proof. We define for all positive integers n,

$$T_0(n) = S(n),$$
$$T_1(n) = T_0(n) - T_0(2n),$$
$$T_2(n) = T_1(n) - T_1(3n)$$
$$T_q(n) = T_{q-1}(n) - T_{q-1}(pn),$$

where p is the qth prime. By the definition of the Möbius function μ it can be seen that

$$T_q(n) = \sum_{j=1}^{\infty} \delta_j^q \mu(j) S(jn). \tag{2.4}$$

On the other hand by Lemma 2,

$$T_0(n) = S(n) = \sum_{k=1}^{\infty} a_{kn},$$

and we can find $T_q(n)$ in terms of the a_{kn}. In order to do this we define α_k^q for k and q positive integers by

$\alpha_k^q = 0$, if k contains one of the first q primes as a prime factor;
$\alpha_k^q = 1$, otherwise.

Then it can be seen that

$$T_q(n) = a_n + \sum_{k=2}^{\infty} \alpha_k^q a_{kn}. \tag{2.5}$$

The theorem will follow from Eqs. (2.4) and (2.5) if we show that

$$\lim_{q \to \infty} \sum_{k=2}^{\infty} \alpha_k^q a_{kn} = 0.$$

But if p is the qth prime we have the estimate,

$$\left| \sum_{k=2}^{\infty} \alpha_k^q a_{kn} \right| \leq \sum_{k=p+1}^{\infty} |a_{kn}|,$$

and the required result follows by the convergence of $\sum_{k=2}^{\infty} |a_{kn}|$.

EXAMPLE. If $q = 3$ the expression for a_1 is

$$a_1 = S(1) - S(2) - S(3) + S(6) - S(5) + S(10) + S(15) - S(30).$$

The terms are written as they arise for $q = 1, 2, 3$ and the terms for which $\mu(j) = 0$ have been omitted. The evaluation requires 30 equally spaced points. Generally summability by primes reduces the necessity for interpolation between sample points and gives numerical answers which are comparable with those obtained by interpolation methods (see [14]).

3. Reduction of the Standard Algorithm

The reduction of the standard algorithm which we are about to describe, depends on the result that for every positive integer a,

$$\sum_{\substack{k=1\\(k,a)=1}}^{\infty} \frac{\mu(k)}{k} = 0.$$

If $a = 1$ this is a theorem of E. Landau which is equivalent to the prime number theorem, [3]. For general a, estimates are obtained for the partial sums in [2, p. 319]. The theorem which we shall require for the reduction is the following.

THEOREM 2. *For each positive integer a*

$$\sum_{j=1}^{\infty} \frac{\mu(ja)}{ja} = 0.$$

Proof. The proof is immediate since,

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{\mu(ja)}{ja} &= \sum_{\substack{k=1\\(k,a)=1}}^{\infty} \frac{\mu(ka)}{ka}\\ &= \frac{\mu(a)}{a} \sum_{\substack{k=1\\(k,a)=1}}^{\infty} \frac{\mu(k)}{k}\\ &= 0.\end{aligned}$$

The simplest reduction of the standard algorithm can be obtained by observing that the coefficient of $f(0)$ in a_1, is $\sum_{k=0}^{\infty} \mu(k)/k = 0$. We now define a new algorithm by

$$a_1 = \sum_{k=1}^{\infty} \mu(k) S(k) - \sum_{k=1}^{\infty} \frac{\mu(k)}{k} f(0),$$

in which the sample value $f(0)$ is deleted. The principal idea of this section may now be introduced. It is simply that the algorithm given by (1.1) and (1.2) can be reduced by the deletion of any finite subset of sample points.

Suppose that we consider Eqs. (1.1) and (1.2) for general n and suppose that a particular sample point *first* occurs in $S(na)$. That is, for $k = a$ as k varies in the summation in (1.1). This sample point then appears precisely in the terms $S(naj)$, j a positive integer, with coefficient

$$c = \sum_{j=1}^{\infty} \frac{\mu(aj)}{naj} = 0.$$

Clearly the point may be deleted from the algorithm (1.1) by the subtraction of the infinite series corresponding to the coefficient c. The procedure can be repeated for any finite subset of sample points.

4. Reduction of the Summability Algorithm

The reduction of the summability by primes algorithm depends on the next theorem which is the analogue of Theorem 2.

Theorem 3. *For each positive integer a*

$$\lim_{q \to \infty} \sum_{j=1}^{\infty} \delta_j^q \frac{\mu(ja)}{ja} = 0.$$

Proof. We first prove the result for the special case $a = 1$. Let $r = p_1 p_2 \ldots p_q$ be the product of the *first* q primes and let $\Sigma_{d/r}$ denote a summation over all divisors of r. Then by omitting the terms for which $\mu(j) = 0$, we have the decomposition,

$$\begin{aligned}\sum_{j=1}^{\infty} \delta_j^q \frac{\mu(j)}{j} &= \sum_{d/r} \frac{\mu(d)}{d} \\ &= \prod_{i=1}^{q} \left(1 - \frac{1}{p_i}\right).\end{aligned}$$

The special case $a = 1$ now follows since the infinite product diverges to zero (see the Mertens theorem [6] for asymptotic estimates).

In the general case it suffices to consider a positive integer a with a prime factorisation $a = p_1 p_2 \ldots p_N$ containing N distinct primes. Let the largest of these primes be the Mth prime amongst *all* primes and let $\sum_{d/a}$ denote a summation over all divisors of a. If $q > M$, then $\delta_{kd}^q = \delta_k^q$ and we have the decomposition

$$\sum_{j=1}^{\infty} \delta_j^q \frac{\mu(j)}{j} = \sum_{d/a} \sum_{\substack{k=1 \\ (k,a)=1}}^{\infty} \delta_k^q \frac{\mu(kd)}{kd}$$

$$= \left[\sum_{d/a} \frac{\mu(d)}{d}\right] \sum_{\substack{k=1 \\ (k,a)=1}}^{\infty} \delta_k^q \frac{\mu(k)}{k}$$

$$= \left[\prod_{i=1}^{N} \left(1 - \frac{1}{p_i}\right)\right] \sum_{\substack{k=1 \\ (k,a)=1}}^{\infty} \delta_k^q \frac{\mu(k)}{k}.$$

If we now take $\lim_{q\to\infty}$, we have by the special case $a = 1$,

$$\lim_{q\to\infty} \sum_{\substack{k=1 \\ (k,a)=1}}^{\infty} \delta_k^q \frac{\mu(k)}{k} = 0.$$

The general case will now follow since

$$\lim_{q\to\infty} \sum_{j=1}^{\infty} \delta_j^q \frac{\mu(ja)}{ja} = \lim_{q\to\infty} \sum_{\substack{k=1 \\ (k,a)=1}}^{\infty} \delta_k^q \frac{\mu(ka)}{ka}$$

$$= \frac{\mu(a)}{a} \lim_{q\to\infty} \sum_{\substack{k=1 \\ (k,a)=1}}^{\infty} \delta_k^q \frac{\mu(k)}{k}$$

$$= 0.$$

Any finite set of sample points may now be deleted from the summability algorithm. The deletion proceeds as in Section 3.

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